Q. $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \cdots + n^2 \cdot C_n =$

i.e. $\sum r^2 \cdot C_r$

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n$$

$$n(1+x)^{n-1} = C_1 + 2 \cdot C_2 x + 3 \cdot C_3 x^2 + \cdots + n \cdot C_n x^{n-1}$$

Again diff.

$$n(n-1)(1+x)^{n-2} = 1 \cdot 2 \cdot C_2 + 2 \cdot 3 \cdot C_3 x + 3 \cdot 4 \cdot C_4 x^2 + \cdots (n-1) n \cdot C_n x^{n-2}$$

$$n(n-1)\cdot 2^{n-2} = 1\cdot 2\, C_2 + 2\cdot 3\, C_3 + 3\cdot 4\, C_4 + \cdots + \underline{n(n-1)\cdot C_n}$$

$$n(n-1)\cdot 2^{n-2} = \sum_{r=2}^{n} (r-1)\cdot r\cdot C_r = \sum_{r=1}^{n} (r-1) r\cdot C_r = \sum_{r=0}^{n} (r-1)\cdot r\, C_r$$

$$n(n-1)\cdot 2^{n-2} = \sum_{r=0}^{n} \left(r^2\cdot C_r - r\, C_r \right)$$

$$\begin{aligned} & n\cdot 2^{n-1} \\ &= n\cdot 2\cdot 2^{n-2} \\ &= 2n\cdot 2^{n-2} \end{aligned}$$

So, $$\sum_{r=0}^{n} r^2\cdot C_r = n(n-1)\cdot 2^{n-2} + \sum_{r=0}^{n} r\, C_r$$

$$= \underline{n(n-1)\cdot 2^{n-2}} + n\cdot 2^{n-1}$$

$$\sum_{r=0}^{n} r^2 \cdot {}^nC_r = n(n-1)\cdot 2^{n-2} + 2n\cdot 2^{n-2}$$

$$= n\cdot 2^{n-2}\left[n-1+2\right]$$

$$\boxed{\sum r^2 \cdot {}^nC_r = n(n+1)\cdot 2^{n-2}}$$

$3, 7, 11, 15 \ldots$

$3 + (n-1)4$

$4n - 1 + 4 = 4n+3$

Q. Evaluate $1\cdot 3\, C_0 + 3\cdot 7\cdot C_1 + 5\cdot 11\cdot C_2 + 7\cdot 15\cdot C_3 + \ldots$ up $(n+1)$ terms

$$\sum_{r=0}^{n} (2r+1)(4r+3)\cdot C_r$$

$$\sum_{r=0}^{n} (2r+1)(4r+3) \cdot C_r$$

$$✓ \quad = \sum_{r=0}^{n} (8r^2+10r+3) \cdot C_r$$

$$= 8\underbrace{\sum_{r=0}^{n} r^2 C_r}_{} + 10\underbrace{\sum_{r=0}^{n} r C_r}_{} + 3\underbrace{\sum_{r=0}^{n} C_r}_{}$$

$$= 8 \cdot n(n+1) \cdot 2^{n-2} + 10 \cdot n \cdot 2^{n-1} + 3 \cdot 2^n$$

Q. Evaluate following

(1) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

$$\int_0^x (1+x)^n = \int_0^x (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n)$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left. C_0 \cdot x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \dots + \frac{C_n \cdot x^{n+1}}{n+1}\right]_0^x$$

$$\frac{(x+1)^{n+1} - 1}{(n+1)} = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \dots + \frac{C_n \cdot x^{n+1}}{n+1} \quad \text{--- (1)}$$

Put $x=1$ in ①

② $$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \frac{C_4}{5} - - - - + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$$

Put $x=-1$ in ① result

$$-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \frac{C_4}{4} - - - = \frac{0-1}{n+1}$$

③ $$3 \cdot {}^{10}C_0 + 3^2 \cdot \frac{{}^{10}C_1}{2} + 3^3 \cdot \frac{{}^{10}C_2}{3} + - - - + 3^{11} \cdot \frac{{}^{10}C_{10}}{11}$$

Put $x=3$ & $n=10$ in result ①

$$\frac{4^{11}-1}{11}$$

Q Prove that

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left\{ \left(\frac{1}{2}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \cdots + m \text{ terms} \right\}$$

$$\sum_{r=0}^{n} {}^nC_r (-1)^r \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} {}^nC_r (-1)^r \left(\frac{3}{4}\right)^r + \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left(\frac{7}{8}\right)^r + \cdots$$

$$\sum_{r=0}^{n} {}^nC_r \left(-\frac{1}{2}\right)^r + \sum_{r=0}^{n} {}^nC_r \left(-\frac{3}{4}\right)^r + \sum_{r=0}^{n} {}^nC_r \left(-\frac{7}{8}\right)^r + \cdots \; m \text{ term}$$

$$(1-1/2)^n + (1-3/4)^n + (1-7/8)^n + \cdots \; m \text{ term}$$

$$1/2^n + 1/4^n + 1/8^n + \cdots$$

$$(1-x)^n = \sum_{r=0}^{n} {}^nC_r (-x)^r$$

general term, ${}^nC_r (-x)^r \cdot 1^{n-r}$

$$= {}^nC_r (-x)^r$$

$$\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n - - -$$

$$\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{2n} + \left(\frac{1}{2}\right)^{3n} - - - \text{ m terms}$$

(G.P.)